Find the general solution of $y^{(4)} - y = 24\sin t$. **SCORE: _____ / 40 PTS**

$r^4 - 1 = 0$

$(r^2-1)(r^2+1)=0$

$r = \pm 1, \pm i$

4 EACH

$y_h = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t$

$y_p = At\cos t + Bt\sin t$

$y_p' = A\cos t - At\sin t + Bt\cos t + B\sin t = (A+Bt)\cos t + (B-At)\sin t$

$y_p'' = B\cos t + (-A-Bt)\sin t + (B-At)\cos t - A\sin t = (2B-At)\cos t + (-2A-Bt)\sin t$

$y_p''' = -A\cos t + (-2B+At)\sin t + (-2A-Bt)\cos t - B\sin t = (-3A-Bt)\cos t + (-3B+At)\sin t$

$y_p^{(4)} = -B\cos t + (3A+Bt)\sin t + (-3B+At)\cos t + A\sin t = (-4B+At)\cos t + (4A+Bt)\sin t$

$y_p^{(4)} - y_p = -4B\cos t + 4A\sin t$

$-4B = 0 \qquad 4A = 24$

$A = 6$

$y = 6t\cos t + C_1 e^t + C_2 e^{-t} + C_3\cos t + C_4 \sin t$

Use elimination (as shown in lecture) to solve the system **SCORE: _____ / 45 PTS**

① $(D+2)[x]+(3D+4)[y]=2t$

② $(2D+3)[x]+(5D+6)[y]=3t$

APPLY $(2D+3)$ TO ①
$-(D+2)$ TO ②

$((2D+3)(3D+4)-(D+2)(5D+6))[y] = 4+6t - (3+6t)$

④ $(D^2+D)[y] = 1$ ④

$r = 0, -1$

$y_h = c_1 + c_2 e^{-t}$

$y_p = At$

$y_p' = A$

$y_p'' = 0$

$y_p'' + y_p' = A = 1$

$y = t + c_1 + c_2 e^{-t}$

② EACH EXCEPT AS NOTED

APPLY $-(5D+6)$ TO ①
$(3D+4)$ TO ②

$(D^2+D)[x] = -(10+12t) + (9+12t)$

$= -1$ ④

$x = -t + k_1 + k_2 e^{-t}$

①

$-1 - k_2 e^{-t}$

$-2\cancel{t} + 2k_1 + 2k_2 e^{-t}$

$3 - 3c_2 e^{-t}$

$4\cancel{t} + 4c_1 + 4c_2 e^{-t} = 2\cancel{t}$

$\longrightarrow (4c_1 + 2k_1 + 2) + (c_2 + k_2)e^{-t} = 0$

$4c_1 + 2k_1 + 2 = 0 \longrightarrow k_1 = -2c_1 - 1$

$c_2 + k_2 = 0 \longrightarrow k_2 = -c_2$

①

$x = -t - (2c_1 + 1) - c_2 e^{-t}$

$y = t + c_1 + c_2 e^{-t}$

THIS IS A TYPO – WAS SUPPOSED TO BE 8

$y = x$ is a solution of $4x^2y'' + 5xy' + y = 6x$.

$y = x^2$ is a solution of $4x^2y'' + 5xy' + y = 19x^2$.

Solve the initial value problem $4x^2y'' + 5xy' + y = 3x - 38x^2$, $y(1) = -1$, $y'(1) = -3$

SCORE: _____ / 25 PTS

$$\hookrightarrow = \tfrac{1}{2}(6x) - 2(19x^2)$$

$$y_p = \underline{\tfrac{1}{2}x} - \underline{2x^2}$$

$\underline{4r^2 + r + 1 = 0}$

③ EACH

$$r = \frac{-1 \pm \sqrt{-15}}{8} = \underline{-\frac{1}{8} \pm \frac{\sqrt{15}}{8} i}$$

$$y = \tfrac{1}{2}x - 2x^2 + \underline{c_1 x^{-\frac{1}{8}} \cos\left(\tfrac{\sqrt{15}}{8} \ln x\right) + c_2 x^{-\frac{1}{8}} \sin\left(\tfrac{\sqrt{15}}{8} \ln x\right)}$$

$$-1 = \tfrac{1}{2} - 2 + c_1 \longrightarrow \underline{c_1 = \tfrac{1}{2}}$$

$$y' = \underline{\tfrac{1}{2} - 4x} \underline{- \tfrac{1}{8} c_1 x^{-\frac{9}{8}} \cos\left(\tfrac{\sqrt{15}}{8} \ln x\right) - \tfrac{\sqrt{15}}{8} c_1 x^{-\frac{9}{8}} \sin\left(\tfrac{\sqrt{15}}{8} \ln x\right)}$$

$$\underline{+ \tfrac{\sqrt{15}}{8} c_2 x^{-\frac{9}{8}} \cos\left(\tfrac{\sqrt{15}}{8} \ln x\right) - \tfrac{1}{8} c_2 x^{-\frac{9}{8}} \sin\left(\tfrac{\sqrt{15}}{8} \ln x\right)}$$

$$-3 = \tfrac{1}{2} - 4 - \tfrac{1}{16} + \tfrac{\sqrt{15}}{8} c_2 \longrightarrow \tfrac{\sqrt{15}}{8} c_2 = \tfrac{9}{16}$$

$$c_2 = \frac{9}{2\sqrt{15}} = \frac{9\sqrt{15}}{30} = \underline{\frac{3\sqrt{15}}{10}}$$

$$\boxed{y = \tfrac{1}{2}x - 2x^2 + \tfrac{1}{2} x^{-\frac{1}{8}} \cos\left(\tfrac{\sqrt{15}}{8} \ln x\right) + \tfrac{3\sqrt{15}}{10} x^{-\frac{1}{8}} \sin\left(\tfrac{\sqrt{15}}{8} \ln x\right)}$$

⑧ EXTRA POINTS DUE TO ADDED COMPLEXITY FROM TYPO

$y = e^{-x}$ is a solution of $xy'' + (2x+1)y' + (x+1)y = 0$. **SCORE: _____ / 40 PTS**

Find the general solution of $xy'' + (2x+1)y' + (x+1)y = \frac{1}{xe^x}$.

$g = \frac{1}{x^2e^x} = x^{-2}e^{-x}$

$y_2 = ve^{-x}$

$y_2' = v'e^{-x} - ve^{-x}$

$y_2'' = v''e^{-x} - 2v'e^{-x} + ve^{-x}$

$$e^{-x}\left(v''x - v'(\cancel{2x}) + \cancel{v(x)} + v'(\cancel{2x}+1) - v(\cancel{2x+1}) + v(\cancel{x+1})\right) = 0$$

(4) $v''x + v' = 0 \rightarrow u = v'$

$\frac{du}{dx} = v''$

(2) EACH EXCEPT AS NOTED

$x\frac{du}{dx} + u = 0$

$\int \frac{1}{u}\,du = \int -\frac{1}{x}\,dx$

$\ln|u| = -\ln|x|$

$v' = u = x^{-1}$

$v = \ln|x|$

$y_2 = e^{-x}\ln|x|$

$$W = \begin{vmatrix} e^{-x} & e^{-x}\ln|x| \\ -e^{-x} & -e^{-x}\ln|x| + x^{-1}e^{-x} \end{vmatrix} = x^{-1}e^{-2x}$$

(4)

$$y_p = -e^{-x}\int \frac{x^{-2}e^{-x}e^{-x}\ln|x|}{x^{-1}e^{-2x}}\,dx + e^{-x}\ln|x|\int \frac{x^{-2}e^{-x}e^{-x}}{x^{-1}e^{-2x}}\,dx$$

$$= -e^{-x}\int \frac{\ln|x|}{x}\,dx + e^{-x}\ln|x|\int \frac{1}{x}\,dx$$

$$= -e^{-x}\cdot\frac{1}{2}(\ln|x|)^2 + e^{-x}(\ln|x|)^2 = \frac{1}{2}e^{-x}(\ln|x|)^2$$

$$y = \frac{1}{2}e^{-x}(\ln|x|)^2 + c_1e^{-x} + c_2e^{-x}\ln|x|$$